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# Freezing transition of the directed polymer in a 1+*d* random medium: Location of the critical temperature and unusual critical properties

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In dimension $d \geqslant 3$, the directed polymer in a random medium undergoes a phase transition between a free phase at high temperature and a low-temperature disorder-dominated phase. For the latter phase, Fisher and Huse have proposed a droplet theory based on the scaling of the free-energy fluctuations $\Delta F(l) \sim l^{\theta}$ at scale $l$. On the other hand, in related growth models belonging to the Kardar-Parisi-Zhang universality class, Forrest and Tang have found that the height-height correlation function is logarithmic at the transition. For the directed polymer model at criticality, this translates into logarithmic free-energy fluctuations $\Delta F_{T_c}(l) \sim (\ln l)^{\sigma}$ with $\sigma = 1/2$. In this paper, we propose a droplet scaling analysis exactly at criticality based on this logarithmic scaling. Our main conclusion is that the typical correlation length $\xi(T)$ of the low-temperature phase diverges as $\ln \xi(T) \sim [-\ln(T_c - T)]^{1/\sigma} \sim [-\ln(T_c - T)]^2$, instead of the usual power law $\xi(T) \sim (T_c - T)^{-\nu}$. Furthermore, the logarithmic dependence of $\Delta F_{T_c}(l)$ leads to the conclusion that the critical temperature $T_c$ actually coincides with the explicit upper bound $T_2$ derived by Derrida and co-workers, where $T_2$ corresponds to the temperature below which the ratio $\overline{Z_L^2}/(\overline{Z_L})^2$ diverges exponentially in $L$. Finally, since the Fisher-Huse droplet theory was initially introduced for the spin-glass phase, we briefly mention the similarities with and differences from the directed polymer model. If one speculates that the free energy of droplet excitations for spin glasses is also logarithmic at $T_c$, one obtains a logarithmic decay for the mean square correlation function at criticality, $\overline{C^2(r)} \sim 1/(\ln r)^{\sigma}$, instead of the usual power law $1/r^{d-2+\eta}$.

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## I. INTRODUCTION

The model of a directed polymer in a random medium plays the role of a "baby spin-glass" model in the field of disordered systems [1–5]. At low temperature, there exists a disorder-dominated phase, where the order parameter is an "overlap" [2,4,6,7]. In finite dimensions, a scaling droplet theory was proposed [5,8], in direct correspondence with the droplet theory of spin glasses [9], whereas in the mean-field version of the model on the Cayley tree, a freezing transition very similar to the one occurring in the random energy model was found [2]. The phase diagram as a function of space dimension $d$ is the following [1]. In dimension $d \leqslant 2$, there is no free phase, i.e., any initial disorder drives the polymer into the strong-disorder phase, whereas for $d > 2$, there exists a phase transition between the low-temperature disorder-dominated phase and a free phase at high temperature [10,11], where the free energy has its annealed value. This phase transition has been studied exactly on a Cayley tree [2] and on hierarchical lattice [12]. In finite dimensions, bounds on the critical temperature $T_c$ have been derived [11,13,14]: $T_0(d) \leqslant T_c \leqslant T_2(d)$. The upper bound $T_2(d)$ corresponds to the temperature above which the ratio $\overline{Z_L^2}/(\overline{Z_L})^2$ remains finite as $L \rightarrow \infty$. The lower bound $T_0$ corresponds to the temperature below which the annealed entropy becomes negative. In $d=3$, the critical properties have been studied numerically by [13,15], with different conclusions. The study of [13] gives a slightly negative value $\alpha \simeq -0.1$, whereas the work of [15] yields a correlation length exponent $\nu \sim 4$, corresponding through hyperscaling to $\alpha = 2 - \nu \sim -2$.

In related growth models belonging to the Kardar-Parisi-Zhang (KPZ) universality class, numerical studies and theoretical arguments [16–18] have found that the height-height correlation function is logarithmic at the transition. For the directed polymer model at criticality, this translates into logarithmic free-energy fluctuations $\Delta F(l) \sim (\ln l)^{\sigma}$ with an exponent $\sigma = 1/2$ that has been measured in $d=3$ [15,16]. In this paper, we make a droplet analysis at criticality based on this logarithmic scaling, in direct correspondence with the Fisher-Huse droplet theory of the low-temperature phase based on the free-energy scaling $\Delta F(l) \sim l^{\theta}$. The matching between the two droplet distributions below $T_c$ and at $T_c$ allows us to derive that the typical correlation length $\xi(T)$ of the low-temperature phase does not follow a power law $(T_c - T)^{-\nu}$, but diverges instead as $\ln \xi(T) \sim [\ln 1/(T_c - T)]^{1/\sigma}$. Moreover, we argue that the logarithmic fluctuations of the free energy at criticality leads to the conclusion that the critical temperature $T_c$ actually coincides with the upper bound $T_2$ derived by Derrida and co-workers, since $T_2$ corresponds to the temperature below which the ratio $\overline{Z_L^2}/(\overline{Z_L})^2$ diverges exponentially in $L$.

The paper is organized as follows. In Sec. II, we explain on the pure wetting and Poland-Scheraga model of DNA denaturation how the transition can be analyzed in terms of the distribution of large loops. The same approach will then be adapted in the remainder of the paper to analyze the freezing transition of the directed polymer, using the loop distribution between two independent copies of the polymer in the same disordered sample. In Sec. III, we describe the Fisher-Huse droplet theory of the low-temperature phase based on the scaling $\Delta F(l) \sim l^{\theta}$. In Sec. IV, we describe the droplet theory based on the logarithmic scaling of the free energy at criticality $\Delta F(l) \sim (\ln l)^{\sigma}$ with $\sigma = 1/2$, and we obtain the divergence of the correlation length $\xi(T)$ near $T_c$, as well as the behavior of the overlap. In Sec. V, we argue that the transition temperature $T_c$ coincides with the upper bound $T_2$ derived by Derrida and co-workers, and discuss why it is not the case in other disordered systems. Finally, in Sec. VI, we discuss the case of spin glasses for which the Fisher-Huse droplet theory was initially developed. If one assumes, by analogy with the directed polymer model, that the free energy of droplet excitations is logarithmic at $T_c$, one obtains some unusual behavior for the correlation function at criticality. Our conclusions are summarized in Sec. VII. In the Appendix, we explain in more detail the matching procedure for the correlation function that we use in this paper.

## II. PURE DELOCALIZATION TRANSITIONS: ANALYSIS IN TERMS OF THE LOOP DISTRIBUTION

### A. Wetting and Poland-Scheraga model of DNA denaturation

The wetting model is defined by the partition function

$$Z_{\text{wetting}}(2L) = \sum_{\text{RW}} \exp\left(\beta \sum_{1 \leq \alpha \leq N} \epsilon_\alpha \delta_{z_{2\alpha},0}\right) \tag{1}$$

where the sum is over one-dimensional random walks (RWs) of $2L$ steps, starting at $z(0)=0$, with increments $z(\alpha+1)-z(\alpha)=\pm 1$. The random walk is constrained to remain in the upper half plane $z \geqslant 0$, but gains an adsorption energy $\epsilon_0$ if $z(\alpha)=0$.

The Poland-Scheraga (PS) model of DNA denaturation [19] is closely related to the wetting model. It describes the configuration of the two complementary strands as a sequence of bound segments and open loops. Each loop of length $l$ has a polymeric entropic weight

$$\mathcal{M}(l) \sim \frac{\mu^l}{l^c} \tag{2}$$

whereas each contact has a Boltzmann weight $e^{-\beta\epsilon_0}$. The partial partition function $Z_{\text{PS}}(1,\alpha)$ with bound ends at monomers 1 and $\alpha$ satisfies the simple recursion relation

$$Z_{\text{PS}}(1,\alpha) = e^{-\beta\epsilon_\alpha} \sum_{\alpha'=1}^{\alpha-1} \mathcal{M}(\alpha-\alpha') Z_{\text{PS}}(1,\alpha'). \tag{3}$$

The wetting model (1) corresponds to a Poland-Scheraga model with parameter $\mu=2$ and loop exponent $c=3/2$ (this exponent comes from the first-return distribution of a one-dimensional random walk). These models without disorder are simple to solve explicitly: they undergo a phase transition between a localized phase at low temperature, characterized by an extensive number of contacts, and a delocalized phase at high temperature. The transition is first order for $c>2$, and second order for $1<c<2$ with correlation length exponent $\nu=1/(c-1)$. Let us now describe how this transition can be understood from the point of view of the loop distribution.

### B. Loop distribution in the low-temperature phase

For the wetting or Poland-Scheraga model with loop exponent $c$, the loop distribution normalized to unity

$$\int dl\, P(l,T) = 1 \tag{4}$$

can be explicitly computed [20]. Near $T_c$, it is useful to decompose it into two terms

$$P(l,T) = P_{\text{finite}}(l,T) + P_{\text{large}}(l,T). \tag{5}$$

The first term represents the statistics of finite loops $l=1,2,\ldots$, whereas the second contribution concerning large loops $l \gg 1$ follows the scaling form

$$dl\, P_{\text{large}}(l,T) = \mathcal{N}(T)\frac{dl}{l}\Phi\left(\frac{l}{\xi(T)}\right). \tag{6}$$

Here $\xi(T)$ is the correlation length that diverges at the transition $\xi(T_c)=\infty$, and the factor $\mathcal{N}(T)$ represents the "normalization" of large loops. In the Poland-Scheraga model, the dependence on $l$ of the probability of a large loop of length $l$ involves the entropic weight $1/l^c$ of a free loop of length $l$ entering the definition of the model (2) and the extensive free-energy cost $\Delta F(l)=\frac{l}{\xi(T)}$ (where we have used hyperscaling)

$$P_{\text{large}}(l,T) \sim \frac{1}{l^c} e^{-\beta l/\xi(T)} \tag{7}$$

so that the scaling function $\Phi$ in Eq. (6) reads

$$\Phi(\lambda) \sim \frac{1}{\lambda^{c-1}} e^{-\beta\lambda}. \tag{8}$$

Let us now consider the number of contacts $n_L(T)$. In the low-temperature phase, it is extensive and simply reads

$$n_L(T) = \frac{L}{\langle b\rangle_T + \langle l\rangle_T} \tag{9}$$

where $\langle b\rangle_T$ represents the averaged length of sequences of consecutive bound monomers (which remains finite as $T \to T_c$), and where $\langle l\rangle_T$ represents the averaged loop length of the full distribution (5)

$$\begin{aligned}\langle l\rangle_T &= \int dl\, l[P_{\text{finite}}(l,T) + P_{\text{large}}(l,T)] \\ &= (\text{finite}) + \int dl\, \mathcal{N}(T)\Phi\left(\frac{l}{\xi(T)}\right) \\ &= (\text{finite}) + \mathcal{N}(T)\xi(T) \int d\lambda\, \Phi(\lambda).\end{aligned} \tag{10}$$

For $1<c<2$, this averaged loop length diverges as

$$\langle l\rangle_T \sim \mathcal{N}(T)\xi(T). \tag{11}$$

In the wetting and Poland-Scheraga models, the energy is directly proportional to the number of contacts (9), so that the energy density also vanishes as

$$e(T) = \frac{1}{\langle l\rangle_T} \sim \frac{1}{\mathcal{N}(T)\xi(T)}. \tag{12}$$

On the other hand, the singularity of the energy is given by the derivative of the free-energy density $f(T) \sim 1/\xi(T)$ with respect to temperature. The critical behavior of the energy is governed by

$$e(T) \sim -\frac{d}{dT}\frac{1}{\xi(T)} \sim \frac{1}{\xi^2(T)}\frac{d\xi(T)}{dT}. \tag{13}$$

The comparison between Eqs. (12) and (13) yields the following differential equation for the correlation length $\xi(T)$ in terms of the normalization $\mathcal{N}(T)$:

$$\frac{d\ln\xi(T)}{dT} = \frac{1}{\mathcal{N}(T)}. \tag{14}$$

### C. Loop distribution at criticality

At criticality, the loop distribution is simply given by the entropic weight that enters the definition of the model (2),

$$dl\, P_{\text{large}}(l,T_c) \sim \frac{dl}{l^c}. \tag{15}$$

For $c>2$, the averaged length $\langle l\rangle = \int dl\, lP_{\text{large}}(l,T_c)$ is finite, so that the number $n(T_c)$ of contacts with the substrate is still extensive $n(T_c)\sim L$ and the transition is first order. For $1<c<2$, the averaged length $\langle l\rangle$ diverges,

$$\langle l\rangle_{T_c} = \int dl\, lP_{\text{large}}(l,T_c) = \infty. \tag{16}$$

The number of contacts is not extensive any longer at criticality $n_L(T_c)/L \sim 1/\langle l\rangle_{T_c} = 0$. Since the Lévy sum of $n$ independent variables $l_i$ drawn from the distribution (15) scales as $l_1+\ldots+l_n \sim n^{1/(c-1)}$, the number of contacts $n_L^{\text{pure}}(T_c)$ in a sample of length $L$ scales as

$$n_L^{\text{pure}}(T_c) \sim L^{c-1} \tag{17}$$

and the transition is second order. Let us now make the connection with the low-temperature phase.

### D. Matching the loop distribution in the critical region

On a length of order $l\sim\xi(T)$, the two expressions (6) and (15) for the loop distribution for $T<T_c$ and for $T=T_c$ should be of the same order,

$$P_{\text{large}}(\xi(T),T) = P_{\text{large}}(\xi(T),T_c). \tag{18}$$

This matching determines the normalization factor $\mathcal{N}(T)$ in terms of the correlation length $\xi(T)$,

$$\mathcal{N}(T) \sim [\xi(T)]^{1-c}. \tag{19}$$

So as the transition is approached, $T\to T_c$, the correlation length diverges, $\xi(T)\to+\infty$, but the density of these large loops vanishes, $\mathcal{N}(T)\to 0$.

The differential equation (14) becomes a closed equation for $\xi(T)$,

$$\frac{d\xi(T)}{dT} \sim [\xi(T)]^c. \tag{20}$$

The integration with the condition $\xi(T_c)=\infty$ gives

$$T_c - T \sim \int_{\xi(T)}^{+\infty}\frac{dx}{x^c} \sim [\xi(T)]^{1-c}, \tag{21}$$

leading to

$$\xi(T) \sim (T_c-T)^{-\nu} \quad \text{with } \nu = \frac{1}{c-1}, \tag{22}$$

in agreement with the exact solution. Note that the normalization of large loops vanishes linearly (19) independently of the loop exponent $c$,

$$\mathcal{N}(T) \sim [\xi(T)]^{1-c} \sim (T_c-T). \tag{23}$$

### E. Important ideas for the following sections on disordered systems

In this section, we have explained how the delocalization transition for the pure wetting or Poland-Scheraga models could be interpreted as a vanishing density $\mathcal{N}(T\to T_c)\to 0$ of large loops of characteristic size that diverges $\xi(T\to T_c)\to\infty$. An important point for the following sections is that the notion of loop distribution which it is natural to consider in the low-temperature phase where the number of contacts is extensive, $n_L(T)\sim L$, has still a meaning exactly at $T_c$ even if the number of contacts is no longer extensive. It loses its meaning only in the high-temperature phase where the number of contacts remains finite. The main idea of the following sections is thus that in disordered systems presenting a low-temperature phase where the order parameter is an overlap, the droplet distribution is not only useful for $T<T_c$ but has still a meaning exactly at $T_c$ even if the overlap vanishes. Moreover, it is precisely this critical droplet distribution that determines the properties of the transition.

## III. DIRECTED POLYMER: LOOP DISTRIBUTION IN THE LOW- TEMPERATURE PHASE

The directed model is defined by the partition function over $d$-dimensional random walks $x(i)$ of $L$ steps,

$$Z_L(\beta) = \sum_{\{x(i)\}} e^{-\beta E(\{x(i\ \})} \quad \text{with } E(\{x(i\ \}) = \sum_{i=1}^{L}\epsilon(i,x(i)) \tag{24}$$

where the random energies $\epsilon(i,x)$ are independent. We have already described in the Introduction its main features, with the corresponding references.

### A. Statistics of excitations above the ground state

The droplet theory for directed polymers [5,8] is similar to the droplet theory of spin glasses [9]. It is a scaling theory that can be summarized as follows. At very low temperature $T\to 0$, all observables are governed by the statistics of low-energy excitations above the ground state. An excitation of large length $l$ costs a random energy

$$\Delta E(l) \sim l^\theta u \tag{25}$$

where $u$ is a positive random variable distributed with some law $Q_0(u)$ having some finite density at the origin $Q_0(u=0)$

$>0$. The exponent $\theta$ is the exponent governing the fluctuation of the energy of the ground state is exactly known in one dimension, $\theta(d=1)=1/3$ [21–24], and for the mean-field version on the Cayley tree, $\theta(d=\infty)=0$ [2]. In finite dimensions $d=2,3,4,5,\ldots$, the exponent $\theta(d)$ has been numerically measured [25–28], and we quote the results of the most precise study we are aware of [28] for dimensions $d=2,3:\theta(d=2)=0.244$ and $\theta(d=3)=0.186$. Note that the existence of a finite upper critical dimension $d_c$ where the exponent would vanish, $\theta(d_c)=0$, has remained a very controversial issue between the numerical studies [25–28] and various theoretical approaches [29–31].

From Eq. (25), the probability distribution of large excitations $l \gg 1$ reads within the droplet theory

$$dl\, \rho(E=0,l) \sim \frac{dl}{l} e^{-\beta \Delta E(l)} \sim \frac{dl}{l} e^{-\beta l^{\theta} u} \tag{26}$$

where the factor $dl/l$ comes from the notion of independent excitations [5]. In particular, its average over the disorder follows the power law

$$dl\, \overline{\rho(E=0,l)} \sim \int_0^{+\infty} du\, Q_0(u) \frac{dl}{l} e^{-\beta l^{\theta} u} = T Q(0) \frac{dl}{l^{1+\theta}}. \tag{27}$$

This prediction describes very well the numerical data in the regime $1 \ll l \ll L$ in dimensions $d=1,2,3$ [32].

### B. Low-temperature phase governed by a zero-temperature fixed point

According to the droplet theory, the whole low-temperature phase $0<T<T_c$ is governed by a zero-temperature fixed point. However, many subtleties arise because the temperature is actually "dangerously irrelevant." The main conclusions of the droplet analysis [5] can be summarized as follows. The scaling (25) governs the free-energy cost of an excitation of length $l$, provided one introduces a correlation length $\xi(T)$ to rescale the length $l$,

$$\Delta F(l) = \left( \frac{l}{\xi(T)} \right)^{\theta} u. \tag{28}$$

Here as before, $u$ denotes a positive random variable distributed with some law $Q(u)$ having some finite density at the origin $Q(u=0)>0$. Moreover, this droplet free energy is a near cancellation of energy and entropy contributions that scale as [5]

$$\Delta E(l) \sim l^{1/2} w, \tag{29}$$

where $w$ is a random variable of order $O(1)$ and of zero mean. The argument is that the energy and entropy are dominated by small-scale contributions of random sign [5], whereas the free energy is optimized on the coarse-grained scale $\xi(T)$. These predictions for the energy and entropy have been numerically checked in [5,33].

### C. Loop distribution for two polymers in the same disordered sample

We now describe how the analysis of Sec. II for the pure transition of the wetting or Poland-Scheraga model can be adapted to the present disordered case. For $T<T_c$, the number of contacts of two independent polymers $x(i)$ and $y(i)$ in the same disordered sample

$$n_L(T) = \sum_{i=1}^{L} \langle \delta_{x(i),y(i)} \rangle \tag{30}$$

is extensive, and the density of contacts, also called the overlap, is precisely the order parameter of the low-temperature phase [2,4,6,7],

$$q(T) = \lim_{L \to \infty} \left( \frac{n_L(T)}{L} \right). \tag{31}$$

Note that on the Cayley tree where $\theta=0$, the distribution of this overlap is made of two $\delta$ peaks at $q=0$ and at $q=1$ [2], whereas in finite dimensions with $\theta>0$, the distribution of this overlap is expected to be a single $\delta$ function at $\overline{q(T)}$ [4].

One may thus analyze the configuration of two polymers in the same sample in terms of contacts separated by loops. Again, it is useful to make the decomposition (5) to concentrate on the contribution of large loops that follows a scaling form based on the free-energy scaling of a droplet of length $l$ (28),

$$dl\, P_{\text{large}}(l,T) = \mathcal{N}(T) \frac{dl}{l} e^{-\beta \Delta F(l)} = \mathcal{N}(T) \frac{dl}{l} e^{-\beta [l/\xi(T)]^{\theta} u}. \tag{32}$$

As in the wetting case, the factor $\mathcal{N}(T)$ represents the normalization of large loops which will be determined below.

The important difference with respect to the pure case is that now the probability of a large loop at a given spatial position depends upon a random variable $u$. This introduces very different behaviors for typical and averaged loop distributions. The averaged loop distribution has the following power-law decay in the whole low-temperature phase:

$$\begin{aligned} dl\, \overline{P_{\text{large}}(l)} &= \frac{dl}{l} \mathcal{N}(T) \int_0^{+\infty} du\, Q(u) e^{-\beta [l/\xi(T)]^{\theta} u} \\ &\underset{l \to \infty}{\simeq} \mathcal{N}(T) Q(0) T \frac{dl}{l} \left( \frac{\xi(T)}{l} \right)^{\theta}, \end{aligned} \tag{33}$$

whereas the typical decay is an exponential with exponent $\theta$,

$$\overline{\ln P_{\text{large}}(l)} = \ln[\mathcal{N}(T)/l] - \beta \left( \frac{l}{\xi(T)} \right)^{\theta} u_0, \tag{34}$$

with $u_0 = \int_0^{+\infty} du\, u Q(u)$. In particular, the typical distribution

$$dl\, P^{\text{typ}}(l) = \mathcal{N}(T) \frac{dl}{l} e^{-\beta [l/\xi(T)]^{\theta} u_0} \tag{35}$$

has a finite first moment

$$\langle l \rangle_{\text{large}} = \int dl\, l P^{\text{typ}}(l) \sim \mathcal{N}(T)\xi(T). \tag{36}$$

As the transition is approached, this term will govern the divergence of the first moment of the full droplet distribution,

$$\langle l \rangle_{\text{tot}} = \langle l \rangle_{\text{finite}} + \langle l \rangle_{\text{large}} \sim \mathcal{N}(T)\xi(T). \tag{37}$$

As a consequence, the number $n_L(T)$ of contacts (and equivalently the number of loops) will have a vanishing density of order

$$\frac{n_L(T)}{L} \sim \frac{1}{\langle l \rangle_T} = \frac{1}{\mathcal{N}(T)\xi(T)}. \tag{38}$$

### D. Energy fluctuations near the transition

Let us now consider the specific heat $c(T)$ that measures the fluctuations of the energy $E_L$,

$$T^2 c(T) = \frac{1}{L}\langle (E_L - \langle E_L \rangle)^2 \rangle = \frac{d^2}{d\beta^2}\left(\frac{\ln Z_L(\beta)}{L}\right). \tag{39}$$

In the high-temperature phase, the specific heat coincides with its annealed value. In particular, for the model (24) with the following Gaussian distribution for the random energies $\epsilon(i,x)$:

$$\rho(\epsilon) = \frac{1}{\sqrt{2\pi}} e^{-\epsilon^2/2}, \tag{40}$$

the annealed specific heat reads

$$T^2 c_{\text{ann}}(T) = 1, \tag{41}$$

meaning that the energy fluctuations per monomer are $\Delta e^2 = 1$. In the low-temperature phase, the free energy is above the annealed free energy, and this implies that the specific heat is less than the annealed value, with a nondiverging singular part [3]. More precisely if one introduces the specific heat exponent $\alpha$, we have

$$c(T) - c_{\text{ann}}(T_c) \sim -A(T_c - T)^{-\alpha} \quad \text{with } A > 0 \text{ and } \alpha \leqslant 0. \tag{42}$$

To interpret this loss of energy fluctuations in the low-temperature phase in terms of the droplets, it is convenient to write the energy fluctuations in terms of the energy difference between two copies $x(i)$ and $y(i)$ of the polymer in the same disordered sample,

$$\begin{aligned}\frac{1}{L}\langle (E_L - \langle E_L \rangle)^2 \rangle &= \frac{1}{2L}\langle [E(\{x(i)\}) - E(\{y(i)\})]^2 \rangle \\ &= \frac{1}{2L}\left\langle \left(\sum_{i=1}^{L} [\epsilon(i,x(i)) - \epsilon(i,y(i))]\right)^2 \right\rangle \\ &= \frac{1}{2L}\left\langle \sum_{i=1}^{L} [\epsilon(i,x(i)) - \epsilon(i,y(i))]^2 \right\rangle \\ &\quad + \frac{1}{L}\left\langle \sum_{1\leqslant i<j\leqslant L} [\epsilon(i,x(i)) - \epsilon(i,y(i))] \times [\epsilon(j,x(j)) - \epsilon(j,y(j))] \right\rangle. \end{aligned} \tag{43, 44}$$

In this sum, the monomers $i$ corresponding to a contact $x(i)=y(i)$ do not contribute, whereas the monomers $i$ inside loops $x(i)\neq y(i)$ are expected to have an energy fluctuation of order $\Delta e_i^2 \sim 1$ [Eq. (29)]. As explained above, for $T \geqslant T_c$ all monomers are characterized by energy fluctuations $\Delta e_i^2 \sim 1$. We thus write that the extensive loss in energy fluctuations just below $T_c$ is proportional to the density of contacts $x(i)=y(i)$ obtained in Eq. (38):

$$\left[\frac{1}{L}\langle (E_L - \langle E_L \rangle)^2 \rangle\right]_{T_c} - \left[\frac{1}{L}\langle (E_L - \langle E_L \rangle)^2 \rangle\right]_T \sim \frac{n_L(T)}{L} \sim \frac{1}{\mathcal{N}(T)\xi(T)}. \tag{45}$$

Near $T_c$, the free-energy density $f(T) = -T \ln Z_L(\beta)/L$ behaves from hyperscaling as

$$f(T) - f(T_c) = \frac{1}{\xi(T)}. \tag{46}$$

Deriving twice with respect to temperature (39), the singular part of the energy fluctuations behaves as

$$\left[\frac{1}{L}\langle (E_L - \langle E_L \rangle)^2 \rangle\right]_{T_c} - \left[\frac{1}{L}\langle (E_L - \langle E_L \rangle)^2 \rangle\right]_T \sim \frac{d^2}{dT^2}\left(\frac{1}{\xi(T)}\right). \tag{47}$$

The consistency of Eqs. (45) and (47) then gives the following differential equation for the correlation length $\xi(T)$ in terms of the normalization $\mathcal{N}(T)$:

$$\frac{d^2}{dT^2}\left(\frac{1}{\xi(T)}\right) \sim \frac{1}{\mathcal{N}(T)\xi(T)}. \tag{48}$$

## IV. DIRECTED POLYMER: LOOP DISTRIBUTION EXACTLY AT CRITICALITY

### A. Logarithmic fluctuations of the free energy

Let us now consider what happens for $T=T_c$. Forrest and Tang [16] have conjectured from their numerical results on a growth model in the KPZ universality class and from the exact solution of another model [34] that the fluctuations of the height of the interface were logarithmic at criticality. For the directed polymer model, this translates into a logarithmic behavior of the free energy fluctuations at $T_c$

$$\Delta F(L,T_c) \sim (\ln L)^{\sigma} v \tag{49}$$

where $v$ is a positive random variable of order 1 distributed with some law $R(v)$, and where the exponent was measured to be in $d=3$ [15,16]

$$\sigma = \frac{1}{2}. \tag{50}$$

Further theoretical arguments in favor of this logarithmic behavior can be found in [17,18]. The argument of [17] is that the power-law behavior $F(L,T_c) \sim L^{\theta_c}$ is impossible at criticality so that $\theta_c=0$. From the scaling relation $\theta_c = 2\zeta_c - 1$ between exponents [35], the roughness exponent $\zeta$ is expected to be exactly $\zeta_c = 1/2$ [17], and a renormalization argument then leads to logarithmic fluctuations of the free energy [18].

### B. Loop distribution at criticality

With the scaling (49), the loop distribution exactly at criticality reads

$$dlP_{T_c}(l) = \frac{dl}{l} e^{-\beta \Delta F(l)} = \frac{dl}{l} e^{-\beta_c (\ln l)^\sigma v}. \tag{51}$$

The averaged loop distribution has then the following extremely slow decay:

$$dl\, \overline{P_{T_c}(l)} = \frac{dl}{l} \int_0^{+\infty} dv\, R(v) e^{-\beta_c (\ln l)^\sigma v} \underset{l \to \infty}{\simeq} R(0) T_c \frac{dl}{l (\ln l)^\sigma} \tag{52}$$

whereas the typical decay is given by

$$\overline{\ln P_{T_c}(l)} = -\ln l - \beta_c (\ln l)^\sigma v_0 \tag{53}$$

with $v_0 = \int_0^{+\infty} dv\, v R(v)$. The typical distribution

$$dl\, P^{\mathrm{typ}}_{T_c}(l) = \frac{dl}{l} e^{-\beta_c (\ln l)^\sigma v_0} \tag{54}$$

has an infinite first moment for $0 < \sigma < 1$ (independently of the prefactor $\beta_c v_0$)

$$\langle l \rangle_{T_c} = \int dl\, l P^{\mathrm{typ}}_{T_c} = \infty \tag{55}$$

and the number of contacts $n_L(T_c)$ is not extensive in $L$.

### C. Matching the typical loop distribution in the critical region

For $l \sim \xi$, the two expressions of the typical correlations in the low-temperature phase (35) and at criticality (54) should coincide as in (18),

$$P_{\mathrm{typ}}(\xi(T), T) = P_{\mathrm{typ}}(\xi(T), T_c). \tag{56}$$

This gives the following relation between the normalization and the correlation length $\xi(T)$,

$$\mathcal{N}(T) \sim e^{-K (\ln \xi)^\sigma}, \tag{57}$$

where $K \sim \beta_c v_0$ is some constant.

Using (48), we thus obtain the following closed differential equation for the correlation length $\xi(T)$:

$$\frac{d^2}{dT^2}\left(\frac{1}{\xi(T)}\right) \sim \frac{1}{\xi(T)} e^{K (\ln \xi)^\sigma}. \tag{58}$$

Near the boundary condition $\xi(T_c) = \infty$, the leading divergence of $\xi(T)$ is given by

$$\frac{1}{(T_c - T)^2} \sim e^{K (\ln \xi)^\sigma}. \tag{59}$$

We thus obtain at leading order that the correlation length $\xi(T)$ diverges as

$$\xi(T) \sim e^{[(2/K) \ln 1/(T_c - T)]^{1/\sigma} + \cdots} \tag{60}$$

instead of the usual power-law behavior $\xi(T) \sim (T_c - T)^{-\nu}$. The free-energy difference thus vanishes very slowly as

$$f(T) - f(T_c) \sim \frac{1}{\xi(T)} \sim e^{-[(2/K) \ln 1/(T_c - T)]^{1/\sigma} + \cdots}. \tag{61}$$

The normalization of large droplet vanishes at leading order as (48)

$$\mathcal{N}(T) \sim (T_c - T)^2 \tag{62}$$

and the contact density or overlap as (31)

$$q(T) = \lim_{L \to \infty} \left( \frac{n_L(T)}{L} \right) \sim \frac{1}{\mathcal{N}(T) \xi(T)} \sim e^{-[(2/K) \ln 1/(T_c - T)]^{1/\sigma} + 2 \ln 1/(T_c - T) + \cdots}. \tag{63}$$

In particular, for the value $\sigma = 1/2$ measured in $d=3$ [15,16], we obtain the final results

$$f(T) - f(T_c) \sim \frac{1}{\xi(T)} \sim e^{-[(2/K) \ln 1/(T_c - T)]^2 + \cdots}, \tag{64}$$

$$q(T) \sim e^{-[(2/K) \ln 1/(T_c - T)]^2 + 2 \ln 1/(T_c - T) + \cdots}. \tag{65}$$

## V. DIRECTED POLYMER: EXPLICIT VALUE OF THE CRITICAL TEMPERATURE

### A. Exact bounds on $T_c$ derived by Derrida and co-workers

Let us first recall the physical meaning of the exact bounds for the critical temperature derived by Derrida and co-workers [11,13,14],

$$T_0(d) \leq T_c \leq T_2(d). \tag{66}$$

The upper bound $T_2(d)$ corresponds to the temperature above which the ratio

$$R_L(T) = \frac{\overline{Z_L^2}}{(\overline{Z_L})^2} \tag{67}$$

remains finite as $L \to \infty$. The lower bound $T_0$ corresponds to the temperature below which the annealed entropy becomes negative.

In dimensions $d=1,2$, the upper bound is at infinity $T_2 = \infty$, whereas for $d \geq 3$, the upper bound $T_2$ is finite. The interpretation is as follows [14]. The ratio (67) can be decomposed according to the probability $P_L(m)$ that two independent usual random walks in dimension $d$ meet $m$ times before time $L$

$$R_L(T) = \sum_{m=1}^{L} P(m) B^m, \tag{68}$$

where the factor

$$B(T) = \frac{\overline{e^{2\beta\epsilon}}}{(\overline{e^{\beta\epsilon}})^2} \tag{69}$$

can be explicitly computed for any distribution of the site disorder variable $\epsilon$. In dimensions $d=1,2$, two random walks meet an infinite number of times as $L \to \infty$, whereas for $d \geqslant 3$, they meet a finite number $m$ of times as $L \to \infty$. The distribution of $m$ decays exponentially

$$P(m) \sim (1-A)A^m \tag{70}$$

where $(1-A)$ is the finite probability of never meeting again. $T_2$ is defined as the temperature where

$$AB(T_2) = 1. \tag{71}$$

For $T > T_2$, $B(T) < B(T_2) = 1/A$, and the ratio $R_L(T)$ has a finite limit

$$R_\infty(T > T_2) = \frac{1-A}{1-AB(T)}. \tag{72}$$

For $T < T_2$, $R_L(T)$ is a geometric series of parameter $AB(T) > 1$, and it thus diverges exponentially in $L$

$$R_L(T < T_2) \sim (1-A)\sum_{m=1}^{L} [AB(T)]^m \sim [AB(T)]^L. \tag{73}$$

Exactly at $T_2$, the ratio diverges but not exponentially,

$$R_L(T_2) = (1-A)\sum_{m=1}^{L} 1 \sim L. \tag{74}$$

### B. Interpretation in terms of the probability distribution of free energies

Let us now interpret the above results of the ratio $R_L(T)$ in terms of the probability distribution $P_L(F)$ of the free energy $F = -kT \ln Z_L$ over the samples of length $L$. By definition (67), one has

$$R_L(T) = \frac{\int dF\, P_L(F) e^{-2\beta F_L}}{\left[\int dF\, P_L(F) e^{-\beta F_L}\right]^2}. \tag{75}$$

For $T > T_2(d)$, the ratio $R_\infty(T)$ is finite: this means that the fluctuations of the free energy over the samples,

$$[\Delta F_L]^2_{\text{samples}} = \int dF\, P_L(F) F^2 - \left(\int dF\, P_L(F) F\right)^2, \tag{76}$$

remain of order $O(1)$ in the limit $L \to \infty$.

On the other hand, for the directed polymer in the low-temperature phase $T < T_c$, the fluctuation of free energies over the samples is expected to have the same scaling as the fluctuation of free energies within the same sample when the end point varies [5]: the fluctuations of free energy over the samples are thus governed by the droplet exponent $\theta$,

$$[\Delta F_L]_{\text{samples}}(T < T_c) \sim [\Delta F_L]_{\text{droplet}}(T < T_c) \sim L^\theta. \tag{77}$$

Let us now recall Zhang's argument [1] that allows one to determine the exponent $\eta$ of the tail of the free-energy distribution,

$$P_L(F \to -\infty) \sim e^{-(|F|/L^\theta)^\eta}. \tag{78}$$

Moments of the partition function can be then evaluated by the saddle-point method, with a saddle value $F^*$ lying in the negative tail (78),

$$\overline{Z_L^n} = \int dF\, P_L(F) e^{-n\beta F_L} \sim \int dF\, e^{-(|F|/L^\theta)^\eta} e^{-n\beta F_L}$$
$$\sim e^{c(n) L^{\theta\eta/(\eta-1)}}. \tag{79}$$

Since these moments of the partition function have to diverge exponentially in $L$, the exponent $\eta$ of the tail (78) reads in terms of the droplet exponent

$$\eta = \frac{1}{1-\theta}. \tag{80}$$

### C. Why $T_c$ coincides with $T_2$ for the directed polymer in finite dimensions

At $T_c$, the fluctuations of the free energy are expected to be logarithmic, as discussed around Eq. (49),

$$\Delta F \sim (\ln L)^\sigma \quad \text{with } \sigma = \frac{1}{2}. \tag{81}$$

From these logarithmic fluctuations, it seems rather difficult to obtain an exponential divergence in $L$ of the ratio $R_L(T_c)$ (75), so the strict inequality $T_c < T_2$ seems very unlikely. On the contrary, if $T_c = T_2$, it is very natural to obtain the divergence found for the ratio at $T_2$ (74),

$$R_L(T_2) \sim L \sim e^{\ln L}. \tag{82}$$

Moreover, to obtain the linear divergence (74), the saddle-point method described above for the low-temperature phase (79) gives that the tail of the free-energy distribution should be at criticality

$$P_{T_c}(F \to -\infty) \sim e^{-[|F|/(\ln L)^\sigma]^{\eta_c}} \quad \text{with } \eta_c = \frac{1}{1-\sigma}. \tag{83}$$

The value $\sigma = 1/2$ corresponds to the tail exponent $\eta_c = 2$.

Our conclusion is thus that the critical temperature in finite dimension $d$ coincides with the temperature $T_2(d)$,

$$T_c(d) = T_2(d). \tag{84}$$

Explicit expressions for $T_2(d)$ in terms of the usual integrals appearing in the theory of random walks can be found in [11,13] for site and bond disorder, respectively.

As a final remark, let us mention that for the directed polymer on the Cayley tree that plays the role of a mean-field version of the model, the critical temperature does not coincide with the upper bound $T_2$, but coincides with the lower bound $T_0$ (66) below which the annealed entropy becomes negative [11],

$$T_c(\text{Cayley}) = T_0(\text{Cayley}) < T_2(\text{Cayley}). \tag{85}$$

This shows that the mean-field limit of the tree structure changes the nature of the transition with respect to the finite-dimensional case. The technical reason seems to be that $\theta=0$ and $\Delta F = O(1)$ in the low-temperature phase of the Cayley tree, whereas Zhang's argument above is consistent only if $\eta = 1/(1-\theta) > 1$ to ensure the convergence in the presence of the exponential term $e^{-\beta n F}$ [Eq. (79)]. When $\theta=0$, the tail of $P_L(F \to -\infty)$ is also an exponential $e^{aF/(\Delta F)}$ as in the random energy model [2] and one has to take into account the minimal free energy that can be obtained for a finite size $L$. From a physical point of view, the reason is that the configurations of two polymers in the same disordered sample are very different. In finite dimensions, contacts and loops alternate extensively, whereas on the tree, the loops simply do not exist: the two polymers may only coincide over some distance and then never meet again. Since the exponential tail found for the Cayley tree actually corresponds to the universal Gumbel tail for the minimum of independent variables, this shows that the nonexponential tail found in finite dimensions reflects the importance of correlations between the free energies of paths due to the presence of loops.

### D. Why $T_c$ is different from $T_2$ in other finite-dimensional disordered systems

The fact that the fluctuations of free energies over the samples have the same scaling as the droplet excitations within one given sample (77) is very specific to the directed polymer model. In other disordered models, such as spin glasses for instance, the fluctuations of free energies over the samples scale instead as [36,37].

$$[\Delta F_L]_{\text{samples}} \sim L^{d/2} \tag{86}$$

at any temperature. This scaling simply reflects the central-limit fluctuations of the $L^d$ disorder variables defining the sample. The directed polymer escapes from these normal fluctuations because it is a one-dimensional path existing in a $1+d$ disordered sample: each configuration of the polymer only experiences $L$ random variables among the $L^{1+d}$ disorder variables that define the sample, and the polymer can "choose" the random variables it sees. So for other disordered systems having fluctuations over the samples governed by (86), the ratio $R_L(T)$ will diverge exponentially as any temperature. The temperature $T_2$ is thus infinite,

$$T_2 = \infty, \tag{87}$$

and has nothing to do with any critical temperature. However, the droplet exponent $\theta$ is expected to govern the correction to the extensive part of the mean value [37],

$$\overline{F_L} \sim L^d f_0 + L^\theta f_1. \tag{88}$$

It can for instance be measured in the free-energy difference upon a change of boundary conditions that forces the introduction of some domain wall in the sample [9].

## VI. SPIN GLASSES

In this section, we briefly mention the analogies and differences between spin glasses and the directed polymer model described above.

### A. Fisher-Huse droplet theory of the low-temperature phase

Discussions on the droplets statistics in pure Ising models can be found in Refs. [38–40]. Here we only summarize very briefly the Fisher-Huse droplet theory of the spin-glass phase [9].

The free energy to make an excitation of large size $l$ follows the scaling

$$\Delta F(l,T) \sim \left(\frac{l}{\xi(T)}\right)^\theta u \tag{89}$$

where $u$ is a positive random variable of distribution $Q(u)$.

The truncated correlation function

$$C(r) \sim |\langle S_0 S_r\rangle - \langle S_0\rangle\langle S_r\rangle| \tag{90}$$

for large distance $r$ is governed by the probability that the two points $o$ and $r$ belongs to the same large droplet

$$C(r) \sim \mathcal{N}(T) e^{-[r/\xi(T)]^\theta u} \tag{91}$$

where the prefactor $\mathcal{N}(T)$ corresponds to the Edwards-Anderson (EA) parameter

$$\mathcal{N}(T) \sim C(\xi(T)) \sim q_{\text{EA}}(T) = \frac{1}{V}\sum_i \langle S_i\rangle^2. \tag{92}$$

The typical decay of the correlation function is

$$\overline{\ln C(r)} = \ln q_{\text{EA}}(T) - \left(\frac{r}{\xi(T)}\right)^\theta u_0, \tag{93}$$

whereas the mean-square correlation function

$$\overline{C^2(r)} \sim T Q(0) q_{\text{EA}}^2(T) \left(\frac{\xi(T)}{r}\right)^\theta \tag{94}$$

has a power-law decay in the whole low-temperature phase.

### B. Spin glasses at criticality

As mentioned above, the exponent $\theta$ of excitations is expected to govern the correction to extensively of the averaged free energy in the low-temperature phase [Eq. (88)]. At criticality where the extensive part of the free energy vanishes, the averaged free energy is thus expected to be governed by the free-energy scale of excitations at $T_c$

$$\overline{F_L}(T_c) \sim [\Delta F_L(T_c)]_{\text{droplet}}. \tag{95}$$

According to the argument of [17], a power-law behavior

$$\overline{F_L}(T_c) \sim L^{\theta_c} \tag{96}$$

is not possible at a critical point, so that $\theta_c = 0$. The droplet free energy at criticality can thus only be logarithmic,

$$[\Delta F_L(T_c)]_{\text{droplet}} \sim (\ln L)^\sigma v, \tag{97}$$

where $v$ is a random variable. In pure models, the power-law decay of truncated correlation can be interpreted as a logarithmic cost with $\sigma_{pure} = 1$. It therefore seems natural to expect $0 < \sigma < 1$, since in the low-temperature phase, the free-energy cost $l^\theta$ in the presence of disorder is much less than the free-energy cost $l^{d-1}$ of pure systems.

Then the truncated correlation (90) is expected to behave at large distance as

$$C(r) \sim e^{-\beta_c \Delta F_r(T_c)} \sim e^{-\beta_c (\ln r)^\sigma v}. \tag{98}$$

The typical decay is then given by

$$\overline{\ln C(r)} = -(\ln r)^\sigma \beta_c v_0 \tag{99}$$

whereas the mean-square correlation function decays very slowly as

$$\overline{C^2(r)} \sim \frac{T_c Q(0)}{(\ln r)^\sigma}, \tag{100}$$

in contrast with the usual power-law correlation $\overline{C^2(r)} \sim 1/r^{d-2+\eta}$ [9]. The matching at scale $r \sim \xi(T)$ between the typical correlation functions at $T < T_c$ [Eq. (93)] and at $T_c$ [Eq. (99)] yields the following relation between the Edwards-Anderson parameter $q_{\text{EA}}(T)$ and the correlation length $\xi(T)$:

$$\ln q_{\text{EA}}(T) \sim -[\ln \xi(T)]^\sigma. \tag{101}$$

For the directed polymer model, it was possible to carry the discussion further because the one-dimensional structure provides a connection between the divergence of large loops and the total number of small loops (38). In spin glasses, the relation between properties of large droplets that dominate the correlation function and properties of small droplets that dominate the energy fluctuations depends on geometric assumptions on the shape of the droplets and on their spatial organization. This goes beyond the present work.

## VII. CONCLUSIONS AND PERSPECTIVES

In this paper, we have proposed a coherent picture of the freezing transition of directed polymers in dimension $d \geq 3$ from the following building blocks: (i) the Fisher-Huse droplet theory of the low-temperature phase; (ii) the Forrest-Tang result concerning the height-height correlation exactly at criticality in related growth models belonging to the KPZ universality class; and (iii) the exact bounds on the critical temperature of Derrida and co-workers.

Our main conclusions are that the critical temperature $T_c$ coincides with the upper bound $T_2$ derived by Derrida and co-workers, and that the logarithmic fluctuations at criticality $\Delta F(l, T_c) \sim (\ln l)^\sigma$ with $\sigma = 1/2$ lead to unusual critical properties. In particular, the typical correlation length $\xi(T)$ of the low-temperature phase, diverges as $\ln \xi(T) \sim [-\ln(T_c - T)]^{1/\sigma} \sim [-\ln(T_c - T)]^2$, instead of the usual power law $\xi(T) \sim (T_c - T)^{-\nu}$. These results emerge from the following picture. Below $T_c$, the number of contacts of two polymers in the same disordered sample is extensive, and their configuration can be thus described as a sequence of contacts and loops. As the transition is approached, the averaged length loops diverges, but our main point is that the loop distribution has still a meaning exactly at $T_c$. (It loses its meaning only in the high-temperature phase.) Moreover, it is precisely this critical droplet distribution that determines the properties of the transition.

Finally, since the directed polymer plays the role of a baby-spin-glass model, we have briefly mentioned some similarities and differences with the directed polymer model. We have discussed some consequences for the Edwards-Anderson order parameter and the mean-square averaged correlation, if one speculates that the free energy of droplet excitations for spin glasses is also logarithmic at $T_c$. Note that the freezing transition of the directed polymer model is completely asymmetric with respect to $T_c$: there is no singularity in thermodynamic quantities as $T \to T_c^+$ [3], since the free energy coincides with its annealed value for $T \geq T_c$. For the spin-glass transition, this raises the question of the relations between the critical properties below and above $T_c$.

We are presently studying numerically various aspects of the freezing transition of the directed polymer in $d = 3$ [41] to see if the scenario proposed in the present paper can be discriminated from the usual power-law critical behaviors used previously in the literature to analyses the data [13,15].

## APPENDIX: MATCHING PROCEDURE FOR CORRELATION FUNCTIONS IN THE CRITICAL REGION

We explain in more detail in this appendix the matching procedure used in the text. We illustrate it with explicit examples concerning the truncated correlation function at large distance

$$C_T(r) = \langle S_0 S_r \rangle - \langle S_0 \rangle \langle S_r \rangle \tag{A1}$$

in pure spin models.

### 1. Finite-size scaling theory for the correlation function

It is useful to introduce the ratio $\rho = r/\xi(T)$ to rewrite the correlation function as

$$C_T(r) = G\left(r, \rho = \frac{r}{\xi(T)}\right). \tag{A2}$$

At criticality where $\xi(T_c) = \infty$, the correlation function is given by the limit $\rho \to 0$

$$C_{T_c}(r) = G(r, \rho = 0). \tag{A3}$$

In the opposite regime where $\rho \gg 1$, the correlation is expected to become a scaling function of the ratio $\rho$ with a temperature-dependent prefactor $\mathcal{N}(T)$

$$C_T(r) \underset{\rho \gg 1}{\simeq} \mathcal{N}(T) \Phi\left(\rho = \frac{r}{\xi(T)}\right). \tag{A4}$$

The matching procedure between the two regimes $\rho \to 0$ and $\rho \gg 1$ consists in requiring that the two expressions should have the same order of magnitude at the matching value $\rho^* = r^*/\xi(T) \sim 1$: this determines the normalization factor $\mathcal{N}(T)$ as

$$\mathcal{N}(T) \sim C_{T_c}(r^* \sim \xi(T)). \tag{A5}$$

We now illustrate the above scheme with Ising and $XY$ models in $d=2$.

### 2. The Ising model in $d=2$

Exactly at $T_c$, the correlation function is given by the power law of exponent $\eta = 1/4$

$$C_{T_c}(r) \simeq \frac{1}{r^{1/4}}. \tag{A6}$$

For $T > T_c$, the correlation function follows the Ornstein-Zernicke form [40]

$$C_{T>T_c}(r) \simeq \mathcal{N}^+(T)\left(\frac{\xi(T)}{r}\right)^{1/2} e^{-r/\xi(T)} \tag{A7}$$

whereas for $T < T_c$ the exact decay at large distance is [42]

$$C_{T<T_c}(r) \simeq \mathcal{N}^-(T)\left(\frac{\xi(T)}{r}\right)^{2} e^{-2r/\xi(T)} \tag{A8}$$

The physical interpretation of this form in terms of droplets can be found in [38–40]. The matching relation yields

$$\mathcal{N}^{\pm}(T) \sim C_{T_c}(r^* \sim \xi(T)) \sim \frac{1}{\xi^{1/4}(T)}. \tag{A9}$$

Using $\xi(T) \sim |T_c - T|^{-1}$, the normalizations of (A7) and (A8) are in agreement with the exact results of [42].

More generally in spin models in dimension $d$ for $T < T_c$, the normalization $\mathcal{N}^-(T)$ is directly related to the order parameter $m(T) = \langle S_i(T)\rangle$ via

$$\mathcal{N}^-(T) \sim m^2(T). \tag{A10}$$

The decay of the correlation function at criticality defines the exponent $\eta$

$$C_{T_c}(r) \sim \frac{1}{r^{d-2+\eta}}. \tag{A11}$$

The matching procedure (A5) then reads

$$\mathcal{N}(T) \sim C_{T_c}(r^* \sim \xi(T)) \sim \frac{1}{\xi^{d-2+\eta}(T)}. \tag{A12}$$

With Eq. (A10), this matching condition simply corresponds to the usual relation $2\beta = (d-2+\eta)\nu$ between critical exponents.

### 3. The $XY$ model in $d=2$

Since the correlation length is infinite in the whole low-temperature phase ($T \leq T_c$), we consider only the matching procedure for $T > T_c$. Exactly at $T_c$, the correlation function is given by [43,44]

$$C_{T_c}(r) \simeq \frac{(\ln r)^{1/8}}{r^{1/4}}. \tag{A13}$$

For $T > T_c$, the correlation function follows the Ornstein-Zernicke form [45]

$$C_{T>T_c}(r) \simeq \mathcal{N}^+(T)\left(\frac{\xi(T)}{r}\right)^{1/2} e^{-r/\xi(T)}. \tag{A14}$$

The matching procedure (A5) yields

$$\mathcal{N}^+(T) \sim C_{T_c}(r^* \sim \xi(T)) \sim \frac{[\ln \xi(T)]^{1/8}}{\xi^{1/4}(T)} \tag{A15}$$

where the correlation length presents the essential singularity divergence $\xi(T) \sim e^{A/(T_c-T)^{1/2}}$.

### 4. Discussion

The above examples show that whenever the critical correlation function is a pure power law [$C_{T_c}(r) \sim r^{-\eta}$], the off-critical correlation function can be rewritten in the factorized form

$$C_T(r) \simeq C_{T_c}(r)\Psi\left(\rho = \frac{r}{\xi(T)}\right) \tag{A16}$$

that corresponds to the form (A4) in the regime $\rho \gg 1$ with $\Psi(\rho) = \rho^{d-2+\eta}\Phi(\rho)$ and $\mathcal{N}(T) = 1/\xi^{d-2+\eta}(T)$.

However, if $C_{T_c}(r)$ is not a pure power law, as in the $XY$ case, the simple factorization (A16) cannot be written.

The relations with the polymer models discussed in the text are as follows. The loop distribution $P(l,T)$ plays the role of a correlation function.

In the pure Poland-Scheraga model where the loop distribution at criticality is a power law (15), the off-critical loop distribution can be written in the factorized form (7)

$$P_{\text{large}}(l,T) \sim P_{\text{large}}(l,T_c)\Psi\left(\frac{l}{\xi(T)}\right) \tag{A17}$$

as in the Ising model above. In this form, the limit $\xi(T) \to \infty$ can be taken to recover the critical distribution.

In the directed polymer model where the critical loop distribution is not a power law but involves a logarithm (51), the off-critical loop distribution (35) cannot be written in a factorized form involving the critical loop distribution, as in the $XY$ model discussed above. In particular, in the explicit expression (35) with (57), valid in the regime $l \gg \xi(T)$, one cannot take blindly the limit $\xi(T) \to \infty$ to recover the critical loop distribution.

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